AVAILABILITY IN STOCHASTIC MODELS

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This article presents stochastic models and associated tools for the availability assessment of a repairable system. Owing to the limited scope of the article, we concentrate on the classical continuous-time pointwise and asymptotic availabilities: we set $E = \mathcal{U} \cup \mathcal{D}$ with $\mathcal{U} \cap \mathcal{D} = \varnothing$, where $\mathcal{U}$ and $\mathcal{D}$ stand for the up- and down-state sets, respectively; the system state is described by a stochastic process $(X_t)_{t \geq 0}$ with $X_t \in E$, and the point availability and its asymptotic version are defined as

$$A(t) = \mathbb{P}(X_t \in \mathcal{U}) \text{ for } t \geq 0,$$
$$A(\infty) = \lim_{t \to +\infty} A(t).$$

Other notions of availability may be found in the literature, please see ***Point and Interval Availability*** and Ref. 1, with lots of references therein.

The article is divided into three sections. The section titled "Availability in Alternating Renewal Models" is devoted to two-state systems, which are considered to be either up or down, with no in-between states. In this case, the system is commonly modeled by a so-called alternating renewal process, which has been extensively studied in the reliability literature [2,3], see also ***Alternating Renewal Processes***.

The section titled "Availability in Markov and Semi-Markov Models" deals with systems with finitely many possible states: typically, such systems are formed of components, which can be up or down, leading to more or less degraded up- and down-states. In such a context, the most commonly used stochastic model is a jump Markov process with a finite state space, which has been thoroughly studied and broadly used in industry [4], see also the section titled *"Continuous-time Markov Chains (CTMCs)"* in this encyclopedia. Such a model implies that both failure and repair rates should remain constant. This drawback has lead to the development and use of semi-Markov processes, which allow for a little more modeling flexibility [5,6]; see also ***Semi-Markov Processes***. The section titled "Markov Models" deals with jump Markov processes and the section titled "Semi-Markov Models" with semi-Markov processes.

Finally, the section titled "Availability in Regenerative and Markov Regenerative Models" is devoted to more general systems, which present regenerative and Markov regenerative properties, with no restrictive condition on the state space or other: between (Markov) regeneration points, the system may have a very general behavior, see [7], ***Regenerative Processes***, and ***Markov Regenerative Processes*** for details. As we shall see, the (Markov) regeneration property then allows to concentrate on the behavior of the system between (Markov) regeneration points, to derive both point and asymptotic availabilities.

In order to illustrate the different stochastic models and associated tools, a small and educational example is used all over the article, under various assumptions: a series system is considered, which is formed of two stochastically independent components A and B, with respective failure and repair rates $(\lambda_A(x), \mu_A(x))$ and $(\lambda_B(x), \mu_B(x))$. (The components' repair durations and times to failure are hence assumed to admit a density with respect to Lebesgue measure). The system always starts from its perfect working state. Owing to its structure, the system is down as soon as one component is down. A repair immediately begins at the failure and puts a component back to its perfect working state (as good as new repairs). According to the cases, the failure of one component

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(and hence of the system) may involve the suspension of the other or not, that is by failure of one component, the other one may go on aging and undertake failure or not, with an eventually reduced failure rate. In each case, both components may be entirely renewed (the repair is then said to be complete) or only the down component. Finally, the failure and repair rates may be constant or general, and the components identical or not.

A very good reference for a deeper insight on the models presented here and for other examples is Ref. 7; for the stochastic tools, see also Refs 5 and 8–10.

Throughout the article, if T stands for a generic random variable (r.v.), $\mathbb{P}_T$ stands for its distribution, $\mathbb{E}(T)$ for its expectation, $F_T(t) = \mathbb{P}(T \le t)$ for its cumulative distribution function, and $\overline{F}_T(t) = \mathbb{P}(T > t) = 1 - F_T(t)$ for its survival function.

AVAILABILITY IN ALTERNATING RENEWAL MODELS

The Example

We assume both components to be identical with a common failure rate $\lambda(x)$ and a repair to be complete, with repair rate $\mu(x)$; we consequently assume that the repair duration always is the same, independent of the degradation state of the system. The length of an up-period is the minimum of two independent r.v.s with common rate $\lambda(x)$ and we have a succession of alternating up- and down independent periods, with respective associate rates $2\lambda(x)$ and $\mu(x)$. The behavior of the system may then be modeled by a so-called alternating renewal process.

The General Case

A system is considered, which evolves according to an alternating renewal process $(X_t)_{t\ge 0}$, where we set $X_t = 1$ when the system is up at time t and $X_t = 0$ when it is down. The system is assumed to start from its perfect working state at time $t = 0$. The successive up-periods are denoted by $U_1, \ldots, U_n, \ldots$ and the down ones by $V_1, \ldots, V_n, \ldots$ Setting $T_n = \sum_{i=1}^{n}(U_i + V_i)$ for $n \ge 1$, the points $T_0 = 0,\ T_1, \ldots,\ T_n, \ldots$ appear as renewal points for the process $(X_t)_{t\ge 0}$. In order to avoid the trivial case $T_0 = T_1 = T_2 = \cdots = 0$ almost surely and following Ref. 5, we assume that $\mathbb{P}(T_1 = 0) < 1$.

With this setting, the system point availability is

$$\begin{aligned} A(t) &= \mathbb{P}(X_t = 1) \\ &= \sum_{n=0}^{+\infty} \mathbb{P}(T_n \le t < T_n + U_{n+1}) \end{aligned}$$

for all $t \ge 0$.

The main tool for its study is the renewal equation (1) fulfilled by $A(t)$, provided just later. To derive it, we classically separate the cases where the first renewal arrives after or before t and we get:

$$\begin{aligned} A(t) &= \mathbb{P}(X_t = 1; t < T_1) + \mathbb{P}(X_t = 1; t \ge T_1) \\ &= \mathbb{P}(t < U_1) \\ &\quad + \int_{[0,t]} \mathbb{P}(X_t = 1 | T_1 = u)\, \mathbb{P}_{T_1}(\mathrm{d}u). \end{aligned}$$

Using the regeneration property at time T_1 (see ***Renewal Function and Renewal-Type Equations*** or Ref. 2 for more details), we easily get

$$\begin{aligned} &\int_{[0,t]} \mathbb{P}(X_t = 1 | T_1 = u)\, \mathbb{P}_{T_1}(\mathrm{d}u) \\ &\quad = \int_{[0,t]} A(t-u)\, \mathbb{P}_{T_1}(\mathrm{d}u) = \left(A * \mathbb{P}_{T_1}\right)(t) \end{aligned}$$

and

$$A(t) = \overline{F}_{U_1}(t) + \left(A * \mathbb{P}_{T_1}\right)(t), \tag{1}$$

where $\mathbb{P}_{T_1}(\mathrm{d}u) = \left(\mathbb{P}_{U_1+V_1}\right)(\mathrm{d}u) = \left(\mathbb{P}_{U_1} * \mathbb{P}_{V_1}\right)(\mathrm{d}u)$ and $*$ stands for the standard convolution.

Apart from very special cases (e.g., exponential distributions), the renewal equation (1) cannot be solved explicitly and numerical procedures have to be developed [11,12].

Limit theorems for solutions of renewal equations (see ***Limit Theorems for Renewal Processes***) allow to get the asymptotic availability from Equation (1): setting $MUT = \mathbb{E}(U_1)$ to be the Mean Up Time of the system on a cycle and $MDT = \mathbb{E}(V_1)$ to be its Mean Down Time, we get:

$$
\begin{aligned}
A(\infty) &= \frac{\displaystyle\int_0^{+\infty} \overline{F}_{U_1}(u)\,\mathrm{d}u}{\mathbb{E}(T_1)} \\
&= \frac{\mathbb{E}(U_1)}{\mathbb{E}(U_1) + \mathbb{E}(V_1)} \\
&= \frac{\mathrm{MUT}}{\mathrm{MUT} + \mathrm{MUT}}, \qquad (2)
\end{aligned}
$$

assuming $\mathbb{E}(U_1) + \mathbb{E}(V_1) < +\infty$ and the distribution of $T_1 = U_1 + V_1$ to be nonlattice [13].

Back to the Example

The times to failure of both components are here assumed to be gamma distributed $\Gamma(a_u, b_u)$, with the following p.d.f. (probability density function):

$$
f_u(x) = \frac{1}{(b_u)^{a_u}\,\Gamma(a_u)} x^{a_u - 1} e^{-x/b_u} \mathbf{1}_{\mathbb{R}_+}(x)
$$

and parameters $(a_u, b_u) = (2, 1/3)$ (mean $= a_u b_u \simeq 0.6667$). The repairs also are gamma distributed with parameters $(a_d, b_d) = (3, 1/36)$ (mean $= a_d b_d \simeq 0.08333$). The point availability has been computed by discretization of Equation (1) and by Monte Carlo simulations. The results of both methods are plotted in Fig. 1, where MC stands for Monte Carlo simulation (5×10^4 histories), MC Sup and MC Inf stand for the upper and lower bounds of the associated 95% confidence band and DM stands for the discretization method. (Similar notations are used all over the article). The asymptotic availability provided by (2) is also indicated in Fig. 1, with $A(\infty) \simeq 0.833$.

In most cases, a system may have several degraded up-states and/or several down-states and cannot hence be modeled by an alternating renewal process. The next section is devoted to such multistate systems, with Markovian or semi-Markovian degradation.

AVAILABILITY IN MARKOV AND SEMI-MARKOV MODELS

Markov Models

The Example: Case of Constant Failure and Repair Rates. We here come back to our example, in the case of different components with constant failure and repair rates ($\lambda_{\mathrm{A}}(x) \equiv \overline{\lambda}_A$, $\mu_{\mathrm{A}}(x) \equiv \overline{\mu}_A$, $\lambda_{\mathrm{B}}(x) \equiv \overline{\lambda}_B$, $\mu_{\mathrm{B}}(x) \equiv \overline{\mu}_B$). In case of failure of one component, the other one is suspended and cannot fail. The evolution of the system may then be modeled by a continuous-time Markov chain (CTMC) $(X_t)_{t \geq 0}$ with range in $E = \{(1,1), (1,0), (0,1)\}$, where the first place refers to component A and the second one to component B, with an "1" indicating an up component, and an "0" a down one.

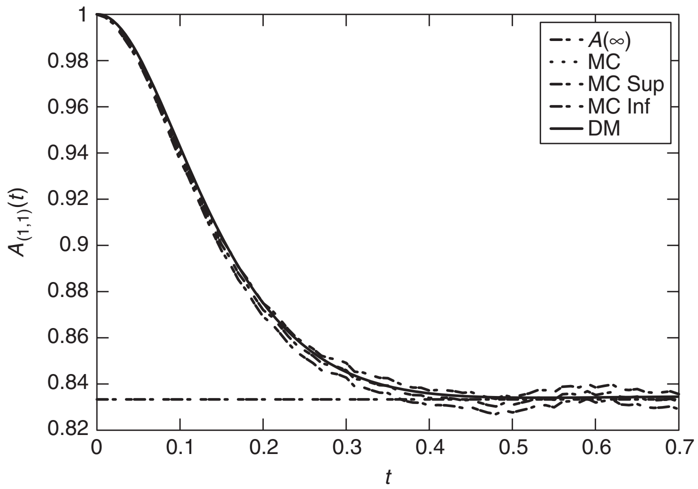

Figure 1. Point and asymptotic availabilities, case of an alternating renewal process.

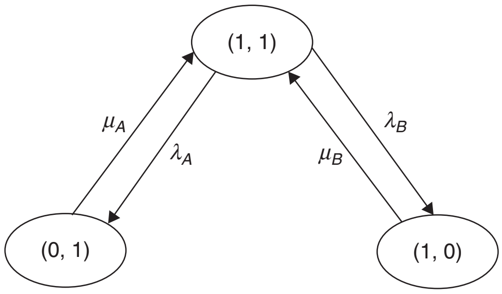

Figure 2. The Markov graph in case of constant failure and repair rates.

The corresponding Markov graph is provided in Fig. 2.

The General Case. In the general case, a system is considered, which evolves in time according to a CTMC $(X_t)_{t\geq 0}$ with range in a finite state space E. We set $\mathcal{A}$ to be the generator matrix of $(X_t)_{t\geq 0}$ and $(P_t)_{t\geq 0}$ to be its transition semigroup:

$$P_t(i,j) = \mathbb{P}_i(X_t = j)$$

for all $i,j \in E$ and all $t \geq 0$, where $\mathbb{P}_i$ is the conditional distribution given that $X_0 = i$, see the section titled *"Continuous-time Markov Chains (CTMCs)"* in this encyclopedia for more details.

The point availability of the system starting from state i is

$$A_i(t) = \sum_{j\in\mathcal{U}} P_t(i,j) = \sum_{j\in\mathcal{U}} \left(e^{t\mathcal{A}}\right)(i,j), \qquad (3)$$

where $e^{t\mathcal{A}}$ refers to the matrix exponentiation.

In case $(X_t)_{t\geq 0}$ is irreducible (and hence recurrent, because of the finite state space), the CTMC $(X_t)_{t\geq 0}$ admits an unique stationary probability measure π, such that $\pi\mathcal{A} = 0$ and $\sum_{i\in E}\pi(i) = 1$, see ***Asymptotic Behavior of Continuous-Time Markov Chains***. The asymptotic availability is then independent of the initial state with

$$A(\infty) = \sum_{i\in\mathcal{U}} \pi(i). \qquad (4)$$

Back to the Example. Owing to the Markov graph provided in Fig. 2, the generator matrix of $(X_t)_{t\geq 0}$ is given by:

$$\mathcal{A} = \begin{pmatrix} -\overline{\lambda}_A - \overline{\lambda}_B & \overline{\lambda}_B & \overline{\lambda}_A \\ \overline{\mu}_B & -\overline{\mu}_B & 0 \\ \overline{\mu}_A & 0 & -\overline{\mu}_A \end{pmatrix}.$$

The point availability of the system is

$$\begin{aligned} A_{(1,1)}(t) &= \mathbb{P}_{(1,1)}(X_t = (1,1)) \\ &= e^{t\mathcal{A}}((1,1),(1,1)). \end{aligned}$$

In case $\overline{\lambda}_A = \overline{\lambda}_B = \overline{\lambda}$ and $\overline{\mu}_A = \overline{\mu}_B = \overline{\mu}$, this easily provides

$$A_{(1,1)}(t) = \frac{\overline{\mu}}{2\overline{\lambda}+\overline{\mu}} + \frac{2\overline{\lambda}}{2\overline{\lambda}+\overline{\mu}} e^{-(2\overline{\lambda}+\overline{\mu})t} \quad \text{and}$$

$$A(\infty) = \frac{\overline{\mu}}{2\overline{\lambda}+\overline{\mu}}.$$

In case of different rates for A and B, we get

$$A(\infty) = \frac{\mu_A\mu_B}{\lambda_A\mu_B + \lambda_B\mu_A + \mu_A\mu_B},$$

and an easy but cumbersome expression for $A_{(1,1)}(t)$, which we do not provide.

Both point and asymptotic availabilities are plotted in Fig. 3 for $\overline{\lambda}_A = 1$, $\overline{\lambda}_B = 2$, $\overline{\mu}_A = 10$ and $\overline{\mu}_B = 15$, with $A(\infty) \simeq 0.811$.

In the Markovian case, both point and asymptotic availabilities hence have an easy expression with respect to the CTMC transition probabilities and stationary probability measure, see Equations (3) and (4). Though the transition probabilities have an explicit expression with respect to the generator matrix (Eq. 3), their numerical evaluation

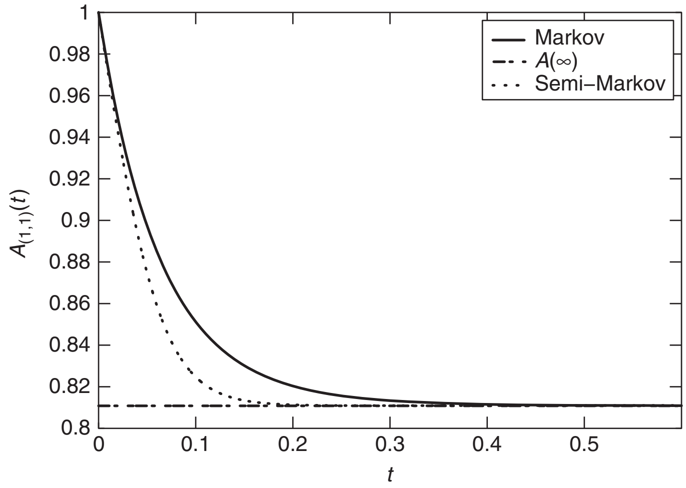

Figure 3. Point and asymptotic availabilities, Markov & semi-Markov cases.

however leads to real difficulties in case of large Markov systems, due to a rapid explosion of the size of the state space with the number of components of the system. This has lead to an extensive literature devoted to their numerical assessment (see ***Computational Methods for CTMCs*** or Refs 14 and 15)

Another drawback of the Markovian models is the underlying assumption of constant failure and repair rates. Such a restrictive assumption may be partially removed by semi-Markov processes, as shown in the next section.

Semi-Markov Models

The Example: Case of Constant Failure Rates and General Repair Rates. We consider here, the same example as in section titled "The Example: Case of Constant Failure and Repair Rates" except the fact that the repair rates are now general [$\mu_{\mathrm{A}}(x)$ and $\mu_{\mathrm{B}}(x)$], while both failure rates remain constant ($\overline{\lambda}_A$ and $\overline{\lambda}_B$). Just as in section titled "The Example: Case of Constant Failure and Repair Rates", no further failure is possible when the system is down.

The possible changes in the system state are due to

- the failure of one component (with the other one suspended in its up-state);
- the end of repair of one component (with the other component up).

It is then easy to see that at each transition time, the system forgets its past; indeed, as the failure rates are constant, the repair of one component (with the other one up) puts the system back to its perfect working state. This means that the system fulfills the Markov property each time its state changes and hence behaves according to a semi-Markov process $(X_t)_{t\geq 0}$ (see ***Semi-Markov Processes***).

The General Case. In the general case, a system is considered, which evolves in time according to a semi-Markov process $(X_t)_{t\geq 0}$ with range in E (finite) and with semi-Markov transition kernel $\left(q\left(i,j,\mathrm{d}t\right)\right)_{i,j\in E}$: we recall that $q\left(i,j,\mathrm{d}t\right) = \mathbb{P}_i\left(X_{T_1} = j, T_1 \in \mathrm{d}t\right)$, where T_1 stands for the first jump time of the process $(X_t)_{t\geq 0}$ (see ***Semi-Markov Processes*** for details). Setting $T_0 = 0 \leq T_1 \leq T_2 \leq \dots$ to be the successive jump times of $(X_t)_{t\geq 0}$, we assume that $\mathbb{P}_i\left(T_0 = T_1 = T_2 = \cdots = 0\right) = 0$ for all $i \in E$, which here again avoids trivialities.

The point availability of the semi-Markov system starting from state i is

$$A_i(t) = \mathbb{P}_i\left(X_t \in \mathcal{U}\right) = \sum_{j\in\mathcal{U}} \mathbb{P}_i\left(X_t = j\right)$$

for all $i \in E$. By separating the cases where the first jump arrives after or before t as in the case of an alternating renewal process, we get

$$\begin{aligned} A_i(t) &= \mathbb{P}_i(X_t \in \mathcal{U}, T_1 > t) + \mathbb{P}_i(X_t \in \mathcal{U}, T_1 \le t) \\ &= \mathbf{1}_{\mathcal{U}}(i)\,\mathbb{P}_i(T_1 > t) \\ &\quad + \sum_{k\in E}\int_{[0,t]} \mathbb{P}_i\left(X_t \in \mathcal{U} | T_1 = u, X_{T_1} = k\right) \\ &\quad \times q\left(i,k,\mathrm{d}u\right) \end{aligned}$$

Applying the Markov property at time T_1, we have

$$\begin{aligned} &\mathbb{P}_i\left(X_t \in \mathcal{U} | T_1 = u, X_{T_1} = k\right) \\ &\quad = \mathbb{P}_k\left(X_{t-u} \in \mathcal{U}\right) = A_k(t-u) \end{aligned}$$

This provides

$$A_i(t) = \mathbf{1}_{\mathcal{U}}(i)\,\mathbb{P}_i(T_1 > t) + (A * q)(i,t), \quad (5)$$

where we set

$$(A * q)(i,t) = \sum_{k\in E}\int_{[0,t]} A_k(t-u)\, q\left(i,k,\mathrm{d}u\right). \quad (6)$$

Equation (5) for $i \in E$ and $t \ge 0$ are known as *Markov renewal equations*, which have no explicit solutions in the general case and have to be solved numerically (see ***Limit Theorems for Markov Renewal Processes*** and Refs 1 and 16).

In the case that the semi-Markov process $(X_t)_{t\ge0}$ is irreducible and that the sojourn times are nonarithmetic with finite means, the process $(X_t)_{t\ge0}$ admits a unique stationary probability measure π, which is known to be identical to the unique stationary probability measure of a CTMC $(Y_t)_{t\ge0}$, which has the same transition matrix $\left(\mathbb{P}_i\left(X_{T_1} = j\right) = \mathbb{P}_i\left(Y_{T_1} = j\right)\right)$ and same mean sojourn times $\mathbb{E}_i(T_1)$ as $(X_t)_{t\ge0}$ (see Refs 5 and 8 for details). This implies that both asymptotic availabilities for the semi-Markov process $(X_t)_{t\ge0}$ and for the Markov process $(Y_t)_{t\ge0}$ are identical.

Back to the Example. The semi-Markovian kernel associated with $(X_t)_{t\ge0}$ is

$$\left(q\left(i,j,\mathrm{d}t\right)\right)_{i,j\in E} = \begin{pmatrix} 0 & \overline{\lambda}_B e^{-(\overline{\lambda}_A+\overline{\lambda}_B)t}\mathrm{d}t & \overline{\lambda}_A e^{-(\overline{\lambda}_A+\overline{\lambda}_B)t}\mathrm{d}t \\ f_{\mu_{\mathrm{B}}}(t)\,\mathrm{d}t & 0 & 0 \\ f_{\mu_{\mathrm{A}}}(t)\,\mathrm{d}t & 0 & 0 \end{pmatrix},$$

where $f_{\mu_{\mathrm{A}}}(t)$ and $f_{\mu_{\mathrm{B}}}(t)$ stand for the respective p.d.f.s associated with r.v.s with respective hazard rates $\mu_{\mathrm{A}}(t)$ and $\mu_{\mathrm{B}}(t)$, where

$$f_{\mu_{\mathrm{A}}}(t) = \mu_{\mathrm{A}}(t)\, e^{-\int_0^t \mu_{\mathrm{A}}(u)\mathrm{d}u},$$

for all $t \ge 0$ and a similar expression for $f_{\mu_{\mathrm{B}}}(t)$, with $\mu_{\mathrm{B}}(t)$ substituted to $\mu_{\mathrm{A}}(t)$.

The Markov renewal equations may here be written as

$$A_{(1,0)}(t) = \int_0^t A_{(1,1)}(t-u)\, f_{\mu_{\mathrm{B}}}(u)\,\mathrm{d}u, \quad (7)$$

$$A_{(0,1)}(t) = \int_0^t A_{(1,1)}(t-u)\, f_{\mu_{\mathrm{A}}}(u)\,\mathrm{d}u, \quad (8)$$

$$\begin{aligned} A_{(1,1)}(t) &= e^{-(\overline{\lambda}_A+\overline{\lambda}_B)t} \\ &\quad + \int_0^t A_{(1,0)}(t-u)\,\overline{\lambda}_B e^{-(\overline{\lambda}_A+\overline{\lambda}_B)u}\mathrm{d}u \\ &\quad + \int_0^t A_{(0,1)}(t-u)\,\overline{\lambda}_A e^{-(\overline{\lambda}_A+\overline{\lambda}_B)u}\mathrm{d}u. \end{aligned} \quad (9)$$

Substituting (7) and (8) in (9) easily provides

$$\begin{aligned} A_{(1,1)}(t) &= e^{-(\overline{\lambda}_A+\overline{\lambda}_B)t} \\ &\quad + \int_0^t A_{(1,1)}(v)\, G(t-v)\,\mathrm{d}v \end{aligned}$$

with

$$\begin{aligned} G(v) &= \int_0^v \left(\overline{\lambda}_B f_{\mu_{\mathrm{B}}}(v-u) + \overline{\lambda}_A f_{\mu_{\mathrm{A}}}(v-u)\right) \\ &\quad \times e^{-(\overline{\lambda}_A+\overline{\lambda}_B)u}\,\mathrm{d}u, \end{aligned}$$

which we solve by discretization. The results are provided in the same figure as for the Markovian case (Fig. 3), with the same constant failure rates ($\overline{\lambda}_A = 1$, $\overline{\lambda}_B = 2$) and gamma repair rates $\Gamma(3, 1/30)$ and

$\Gamma(3.5, 1/52.5)$ with the same means as in the Markovian case from section titled "Back to the Example" in the section titled "Markov Models" (1/10 and 1/15, respectively). As expected, we can observe that the asymptotic availability is the same in both cases, with a slower convergence in the Markovian case.

In a semi-Markov model, the system forgets its past each time it changes state. In the example, assume that the failure rates are not constant any more. In that case, at the end of repair of the down component, the suspended one is not as good as new and it needs to be repaired for the system to be entirely renewed (and for the system to forget its past). If the duration of this repair is independent of the degradation level of the suspended component, the system still is semi-Markovian. However, under the more realistic assumption of a repair depending on the degradation level, this is not true any more. This shows a limitation of the modeling power of the semi-Markov processes. (A classical other limitation is that a parallel two-unit system formed of two independent semi-Markovian components is not semi-Markovian any more). We now come to regenerative and semiregenerative models, which allow for more flexibility.

AVAILABILITY IN REGENERATIVE AND MARKOV REGENERATIVE MODELS

Regenerative Models

The Example. We consider here the case of general failure and repair rates, with suspension of the up component in case of failure. At failure, the system is completely repaired. A single repairman is considered so that the repair durations of both components are added. The repair of one component now depends on its degradation level: for a down component, its repair rate is $\mu_{\mathrm{A}}(x)$ or $\mu_{\mathrm{B}}(x)$, as before. When a component has been functioning for a duration u, the repair rate to bring it back to its perfect working state is some $\tilde{\mu}_A(x,u)$ (or $\tilde{\mu}_B(x,u)$), where $\tilde{\mu}_A(x,u)$ is some decreasing function in u with $\lim_{u\to+\infty}\tilde{\mu}_A(x,u) = \mu_{\mathrm{A}}(x)$ (the same for $\tilde{\mu}_B(x,u)$).

Under the previous assumptions, the state $(1,1)$ is a regenerative state in the sense that each time the system enters this state, it starts again in a similar way as from the beginning and forgets its past. The periods between two successive arrivals in state $(1,1)$ are called *cycles* and the process $(X_t)_{t\geq 0}$ appears as a regenerative one (see ***Regenerative Processes***).

The General Case. In the general case, a system is considered, which evolves in time according to a regenerative process $(X_t)_{t\geq 0}$, with $(T_n)_{n\in\mathbb{N}}$ as regeneration times. Following the definition of Çinlar [5], this means that $(T_n)_{n\in\mathbb{N}}$ are the points of a renewal process such that

1. the T_n's are stopping times adapted to $(X_t)_{t\geq 0}$ (and $\mathbb{P}(T_1 = 0) < 1$);
2. at each T_n, the future process $\left(X_{t+T_n}\right)_{t\geq 0}$ given the past up to T_n (namely given the σ-algebra generated by $\{X_u, u \leq T_n\}$), is identically distributed as $(X_t)_{t\geq 0}$.

See ***Regenerative Processes*** or Ref. 5 for more details. This means that at each T_n, a regenerative system starts again as from the beginning and forgets its past. The evolution of the system between two regeneration points may here be very general.

Using the regeneration property at time T_1, the point availability satisfies the following renewal equation:

$$\begin{aligned} A(t) &= \mathbb{P}(X_t \in \mathcal{U}, T_1 > t) + \mathbb{P}(X_t \in \mathcal{U}, T_1 \leq t) \\ &= \mathbb{P}(X_t \in \mathcal{U}, T_1 > t) \\ &\quad + \int_{[0,t]} A(t-u)\,\mathbb{P}_{T_1}(\mathrm{d}u). \end{aligned} \tag{10}$$

Assuming the distribution of T_1 to be non-lattice and $\mathbb{E}(T_1) < +\infty$, we derive [5]:

$$A(\infty) = \frac{\int_0^{+\infty} \mathbb{P}(X_t \in \mathcal{U}, T_1 > t)\,\mathrm{d}t}{\mathbb{E}(T_1)} = \frac{\mathbb{E}\left(\int_0^{T_1} \mathbf{1}_{\mathcal{U}}(X_t)\,\mathrm{d}t\right)}{\mathbb{E}(T_1)} = \frac{\mathrm{MUT}}{\mathrm{MUT}+\mathrm{MDT}}, \tag{11}$$

where MUT and MDT, are respectively, the cumulated Mean Up Time and Mean Down Time of thex system on a cycle.

Back to the Example. We assume that both times to failure are gamma distributed with the same means as in section titled "Availability in Markov and Semi-Markov Models" (where the failure rates were constant), and we take $\Gamma(2, 1/2)$ and $\Gamma(2, 1/4)$. The repair durations in case of failure are identically distributed as in the semi-Markovian case: $\Gamma\left(a_{R_A}, b_{R_A}\right)$ and $\Gamma\left(a_{R_B}, b_{R_B}\right)$ with $\left(a_{R_A}, b_{R_A}\right) = (3, 1/30)$ and $\left(a_{R_B}, b_{R_B}\right) = (3.5, 1/52.5)$, and respective means $m_{R_A} = a_{R_A} b_{R_A}$ and $m_{R_B} = a_{R_B} b_{R_B}$. When component A has been functioning for u time units, its repair duration is gamma distributed: $\Gamma\left(a_{R_A}, b_{R_A}\left(1 - \frac{1}{1+u^\alpha}\right)\right)$, with mean $m_{R_A}\left(1 - \frac{1}{1+u^\alpha}\right)$, where we assume $\alpha = 1/8$. The distribution of the repair duration is similar for component B, with $\left(a_{R_B}, b_{R_B}\right)$ substituted to $\left(a_{R_A}, b_{R_A}\right)$ and the same α.

A cycle begins with an up-period of length $U_1 = \min\left(Z_{\lambda_\mathrm{A}}, Z_{\lambda_\mathrm{B}}\right)$, where Z_ν stands for an r.v. with hazard rate $\nu(t)$, and where Z_{λ_A} and Z_{λ_B} are independent. (Other natural conditions of independence will be assumed further on, which will not be detailed). Then comes a down-period with length

$$V_1 = \left(Z_{\mu_\mathrm{A}} + Z_{\mu_\mathrm{B}\left(\cdot, Z_{\lambda_\mathrm{A}}\right)}\right)\mathbf{1}_{\left\{Z_{\lambda_\mathrm{A}} < Z_{\lambda_\mathrm{B}}\right\}} + \left(Z_{\mu_\mathrm{B}} + Z_{\mu_\mathrm{A}\left(\cdot, Z_{\lambda_\mathrm{B}}\right)}\right)\mathbf{1}_{\left\{Z_{\lambda_\mathrm{A}} \geq Z_{\lambda_\mathrm{B}}\right\}},$$

where, given Z_{λ_A}, the distribution of $Z_{\mu_\mathrm{B}\left(\cdot, Z_{\lambda_\mathrm{A}}\right)}$ is $\Gamma\left(a_{R_B}, b_{R_B}\left(1 - \frac{1}{1+\left(Z_{\lambda_\mathrm{A}}\right)^\alpha}\right)\right)$ with mean

$$\mathbb{E}\left(Z_{\mu_\mathrm{B}\left(\cdot, Z_{\lambda_\mathrm{A}}(\cdot)\right)} | Z_{\lambda_\mathrm{A}}\right) = m_{R_B}\left(1 - \frac{1}{1+\left(Z_{\lambda_\mathrm{A}}\right)^\alpha}\right). \tag{12}$$

Similarly,

$$\mathbb{E}\left(Z_{\mu_\mathrm{A}\left(\cdot, Z_{\lambda_\mathrm{B}}(\cdot)\right)} | Z_{\lambda_\mathrm{B}}\right) = m_{R_A}\left(1 - \frac{1}{1+\left(Z_{\lambda_\mathrm{B}}\right)^\alpha}\right).$$

Note that though up- and down periods are alternating, their durations are not independent so that an alternating renewal process cannot model the system evolution.

We have

$$\mathbb{P}(X_t \in \mathcal{U}, T_1 > t) = \mathbb{P}(U_1 > t) = \overline{F}_{\lambda_\mathrm{A}}(t)\,\overline{F}_{\lambda_\mathrm{B}}(t),$$

and the relation $T_1 = U_1 + V_1$ then allows to get a discretized version of Equation (10), which we solve numerically. The results are plotted in Fig. 4, as well as those by Monte Carlo simulations.

The asymptotic availability is computed via (11), with

$$\mathrm{MUT} = \mathbb{E}(U_1) = \mathbb{E}\left(\min\left(Z_{\lambda_\mathrm{A}}, Z_{\lambda_\mathrm{B}}\right)\right) = \int_0^{+\infty} \overline{F}_{\lambda_\mathrm{A}}(t)\,\overline{F}_{\lambda_\mathrm{B}}(t)\,\mathrm{d}t \underset{\mathrm{def}}{=} U_{AB}$$

and $\mathrm{MDT} = \mathbb{E}(V_1) = R_{AB} + R_{BA}$, where

$$\begin{aligned}
R_{AB} &= \mathbb{E}\left(\left(Z_{\mu_\mathrm{A}} + Z_{\mu_\mathrm{B}\left(\cdot, Z_{\lambda_\mathrm{A}}\right)}\right)\mathbf{1}_{\left\{Z_{\lambda_\mathrm{A}} < Z_{\lambda_\mathrm{B}}\right\}}\right) \\
&= \mathbb{E}\left(Z_{\mu_\mathrm{A}}\right)\mathbb{P}\left(Z_{\lambda_\mathrm{A}} < Z_{\lambda_\mathrm{B}}\right) \\
&\quad + \mathbb{E}\left(\mathbb{E}\left(Z_{\mu_\mathrm{B}\left(\cdot, Z_{\lambda_\mathrm{A}}\right)}\mathbf{1}_{\left\{Z_{\lambda_\mathrm{A}} < Z_{\lambda_\mathrm{B}}\right\}} | Z_{\lambda_\mathrm{A}}\right)\right) \\
&= m_{R_A}\int_0^{+\infty} f_{\lambda_\mathrm{A}}(u)\,\overline{F}_{\lambda_\mathrm{B}}(u)\,\mathrm{d}u \\
&\quad + \mathbb{E}\left(\mathbb{E}\left(Z_{\mu_\mathrm{B}\left(\cdot, Z_{\lambda_\mathrm{A}}\right)} | Z_{\lambda_\mathrm{A}}\right)\right. \\
&\quad \left.\times \mathbb{E}\left(\mathbf{1}_{\left\{Z_{\lambda_\mathrm{A}} < Z_{\lambda_\mathrm{B}}\right\}} | Z_{\lambda_\mathrm{A}}\right)\right)
\end{aligned}$$

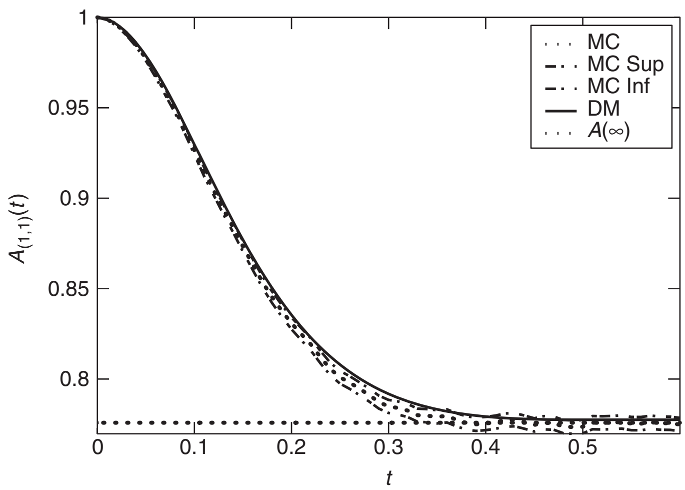

Figure 4. Point and asymptotic availabilities, regenerative case.

$$= m_{R_A}\int_0^{+\infty} f_{\lambda_{\mathrm{A}}}(u)\,\overline{F}_{\lambda_{\mathrm{B}}}(u)\,\mathrm{d}u + \mathbb{E}\left(m_{R_B}\left(1-\frac{1}{1+(Z_{\lambda_{\mathrm{A}}})^{\alpha}}\right)\overline{F}_{\lambda_{\mathrm{B}}}(Z_{\lambda_{\mathrm{A}}})\right) \tag{13}$$

$$= \int_0^{+\infty}\left(m_{R_A}+m_{R_B}\left(1-\frac{1}{1+u^{\alpha}}\right)\right) \times f_{\lambda_{\mathrm{A}}}(u)\,\overline{F}_{\lambda_{\mathrm{B}}}(u)\,\mathrm{d}u$$

using Equation (12) for (13). A similar expression is valid for R_{BA} and we finally have:

$$A(\infty) = \frac{U_{AB}}{U_{AB}+R_{AB}+R_{BA}}$$

where each quantity is now easily computable. This provides: $A(\infty) \simeq 0.776$.

As already mentioned, regenerative processes allow for a great flexibility as for the system behavior between regeneration points. However, the numerical assessment of the asymptotic availability requires the computation of both mean up time and mean down time on a cycle, which is not always that easy, even in rather simple cases. (As for the point availability, it is generally still more complicated). In some cases, the system may forget its past when entering several states and not only one single state, which may lead to a so-called Markov regenerative process. Though most of the Markov regenerative processes used in reliability also are regenerative processes, the Markov regeneration property usually leads to different expressions for the point and/or asymptotic availabilities from those obtained with the simple regeneration property, which may be easier to compute. The discussion on Markov Regenerative models follows.

Markov Regenerative Models

The Example. We first come back to our example in a slightly modified version: both failure rates are constant ($\lambda_{\mathrm{A}}(x) \equiv \overline{\lambda}_A$, $\lambda_{\mathrm{B}}(x) \equiv \overline{\lambda}_B$), and the repair rates are general ($\mu_{\mathrm{A}}(x)$ and $\mu_{\mathrm{B}}(x)$) as in section titled "The Example: Case of Constant Failure Rates and General Repair Rates." However, in case of failure of one component, the other one may now fail while the down component is repaired, at a lower rate however ($\overline{\lambda}_A'$ and $\overline{\lambda}_B'$ with $\overline{\lambda}_A' \leq \overline{\lambda}_A$ and $\overline{\lambda}_B' \leq \overline{\lambda}_B$). There is one single repairman and the repair of one component is completed up to its end, even in case of arrival of a new failure. To model it, we add two new states to our system: state $(0_R, 0)$ which means that component A is down under repair and that component B is down, waiting for a repair; the same for state $(0, 0_R)$.

Entrance in state $(1,1)$ clearly is a regenerative point and $(X_t)_{t\geq 0}$ is a regenerative process. However, the mean length of a cycle, here, is a little complicated by the fact that lots of different scenarios are possible, such as $(1,1) \to (0,1) \to (0_R,0) \to (1,0) \to (0,0_R) \to (0,1) \to \ldots \to (1,1)$. An alternative model is however possible, which leads to simpler computations; indeed, because of the constant failure rates, the system forgets its past each time it enters states $(1,1)$, $(1,0)$ and $(0,1)$. Note that the system, however, does not fulfill the same property when entering states $(0_R,0)$ and $(0,0_R)$. In such cases, the repair of one component has indeed already begun, with a general repair rate. The process $(X_t)_{t\geq 0}$ is consequently not semi-Markovian. One can nevertheless consider a new process $(Y_t)_{t\geq 0}$ defined on a new state space $F=\{1,2,3\}$, which enters state 1, 2, and 3 each time $(X_t)_{t\geq 0}$ enters the states $(1,1)$, $(1,0)$, and $(0,1)$, respectively. The possible transitions for $(X_t)_{t\geq 0}$ from state $(1,1)$ are $(1,1)\to(1,0)$ and $(1,1)\to(0,1)$ with respective rates $\overline{\lambda}_B$ and $\overline{\lambda}_A$. The possible transitions for $(Y_t)_{t\geq 0}$ from state 1 hence are $1\to 2$ and $1\to 3$ with the same respective rates, $\overline{\lambda}_B$ and $\overline{\lambda}_A$. From state $(1,0)$, the process $(X_t)_{t\geq 0}$ may go back to state $(1,1)$ with probability $p_1=\mathbb{P}\left(Z_{\mu_\mathrm{B}}\leq Z_{\overline{\lambda}'_A}\right)$, where $Z_{\overline{\lambda}'_A}$ and Z_{μ_B} stand for independent r.v.s with respective hazard rates $\overline{\lambda}'_A$ and $\mu_\mathrm{B}(t)$. From state $(1,0)$, the process $(X_t)_{t\geq 0}$ may also go to state $(0,0_R)$ and next to state $(0,1)$ with probability $1-p_1$. This means that from state 2, the possible transitions for $(Y_t)_{t\geq 0}$ are $2\to 1$ and $2\to 3$, with respective probabilities p_1 and $1-p_1$. Moreover, the time spent in state 2 by $(Y_t)_{t\geq 0}$ is some r.v. with hazard rate $\mu_\mathrm{B}(t)$. A similar reasoning is valid for state $(0,1)$ and proves that the process $(Y_t)_{t\geq 0}$ is semi-Markovian, with the following semi-Markovian kernel:

$$\left(q\left(i,j,\mathrm{d}t\right)\right)_{i,j\in F}=\begin{pmatrix} 0 & \overline{\lambda}_B e^{-(\overline{\lambda}_A+\overline{\lambda}_B)t}\mathrm{d}t & \overline{\lambda}_A e^{-(\overline{\lambda}_A+\overline{\lambda}_B)t}\mathrm{d}t \\ p_1 f_{\mu_\mathrm{B}}(t)\,\mathrm{d}t & 0 & (1-p_1)f_{\mu_\mathrm{B}}(t)\,\mathrm{d}t \\ p_2 f_{\mu_\mathrm{A}}(t)\,\mathrm{d}t & (1-p_2)f_{\mu_\mathrm{A}}(t)\,\mathrm{d}t & 0 \end{pmatrix},$$

where $p_2=\mathbb{P}\left(Z_{\mu_\mathrm{A}}\leq Z_{\overline{\lambda}'_B}\right)$, same notations, and where $f_{\mu_\mathrm{A}}(t)$ and $f_{\mu_\mathrm{B}}(t)$ have been defined in Subsection 2.2.3. Though the process $(Y_t)_{t\geq 0}$ does not fully describe the system evolution, (the process $(X_t)_{t\geq 0}$ may evolve while $(Y_t)_{t\geq 0}$ remains constant), the process $(X_t)_{t\geq 0}$ forgets its past at each jump of the semi-Markovian process $(Y_t)_{t\geq 0}$. This means that $(X_t)_{t\geq 0}$ is a Markov regenerative process, with $(Y_t)_{t\geq 0}$ as imbedded semi-Markov process (or $\left(T_n,Y_{T_n}\right)_{n\in\mathbb{N}}$ as imbedded Markov renewal process, equivalently).

The General Case. In the general case, a system is considered, which evolves in time according to a Markov regenerative (or semiregenerative) process $(X_t)_{t\geq 0}$, with state space E. Following Ref. 5, this means that $t\to X_t(\omega)$ is right continuous and has left-hand limits for almost all ω and that there exists a Markov renewal process $(T_n,Y_n)_{n\in\mathbb{N}}$ on a state space F (here assumed finite) such that (i) the T_n's are stopping times adapted to $(X_t)_{t\geq 0}$ [with $\mathbb{P}_i(T_0=T_1=T_2=\ldots=0)=0$ for all $i\in F$]; (ii) for each T_n, the r.v. Y_n is measurable with respect to the σ-algebra $\sigma(X_u,u\leq T_n)$ generated by the past up to T_n; (iii) at each T_n, the future process $\left(X_{t+T_n}\right)_{t\geq 0}$ given the past up to T_n (i.e., given $\sigma(X_u,u\leq T_n)$) is identically distributed on $\{Y_n=i\}$ as $(X_t)_{t\geq 0}$ given $\{Y_0=i\}$ (see ***Markov Regenerative Processes*** for more details). In other words, this means that the future of a Markov regenerative process after T_n only depends on Y_n and that Y_n only depends on the past of the process up to T_n. As in the case of a regenerative process, the evolution of the system between two jumps of $(Y_n)_{n\in\mathbb{N}}$ may be very general.

Setting $\left(q\left(i,j,\mathrm{d}t\right)\right)_{i,j\in F}$ to be the semi-Markov kernel associated to $(T_n,Y_n)_{n\in\mathbb{N}}$ and using the Markov property at time T_1, the point availability of the system given that $Y_0=i$ fulfills the following Markov renewal equation:

$$\begin{aligned} A_i(t) &= \mathbb{P}_i(X_t\in\mathcal{U},T_1>t)+\mathbb{P}_i(X_t\in\mathcal{U},T_1\leq t) \\ &= \mathbb{P}_i(X_t\in\mathcal{U},T_1>t) \\ &\quad+\sum_{j\in E}\int_{[0,t]}\mathbb{P}_i(X_t\in\mathcal{U}|T_1=u,Y_1=j) \\ &\quad\times q\left(i,j,\mathrm{d}u\right) \end{aligned}$$

$$= \mathbb{P}_i\left(X_t \in \mathcal{U}, T_1 > t\right) + \sum_{j \in E} \int_{[0,t]} A_j\left(t-u\right) q\left(i,j,\mathrm{d}u\right)$$
$$= \mathbb{P}_i\left(X_t \in \mathcal{U}, T_1 > t\right) + \left(A * q\right)\left(i,t\right), \quad (14)$$

where $\left(A * q\right)\left(i,t\right)$ is defined in Equation (6).

Let us now suppose that the Markov renewal process $\left(T_n, Y_n\right)_{n\in\mathbb{N}}$ is irreducible with nonarithmetic sojourn times and means $m\left(i\right) = \mathbb{E}_i\left(T_1\right)$, and let π be the unique stationary probability measure of the Markov chain $\left(Y_n\right)_{n\in\mathbb{N}}$. The asymptotic availability $A\left(\infty\right)$ then exists and is [5]

$$A\left(\infty\right) = \frac{\sum_{i\in F} \pi\left(i\right) \int_0^{+\infty} \mathbb{P}_i\left(X_t \in \mathcal{U}, T_1 > t\right) \mathrm{d}t}{\sum_{i\in F} \pi\left(i\right) m\left(i\right)}$$
$$= \frac{\sum_{i\in F} \pi\left(i\right) \mathbb{E}_i\left(\int_0^{T_1} \mathbf{1}_{\mathcal{U}}\left(X_t\right) \mathrm{d}t\right)}{\mathbb{E}_\pi\left(T_1\right)} \quad (15)$$

where symbols i and π in $\mathbb{P}_i$, $\mathbb{E}_i$, and $\mathbb{E}_\pi$ here refer to the initial distribution of $\left(Y_n\right)_{n\in\mathbb{N}}$.

This means that the asymptotic availability is the quotient of the system mean up time between arrivals of $\left(T_n, Y_n\right)_{n\in\mathbb{N}}$ divided by the mean interarrival length of $\left(T_n, Y_n\right)_{n\in\mathbb{N}}$, when $\left(T_n, Y_n\right)_{n\in\mathbb{N}}$ is in its steady state.

Back to the Example. We take $\overline{\lambda}_A = 1$, $\overline{\lambda}_B = 2$ as failure rates, and $\Gamma\left(3, 1/30\right)$, $\Gamma\left(3.5, 1/52.5\right)$ for the repair distributions, as in the semi-Markovian case. We also take $\overline{\lambda}'_A = 0.5$ and $\overline{\lambda}'_B = 1$.

The Markov renewal Equations (14) may here be written as

$$A_1\left(t\right) = e^{-\left(\overline{\lambda}_A+\overline{\lambda}_B\right)t} + \int_0^t A_2\left(t-u\right) q\left(1,2,\mathrm{d}u\right) + \int_0^t A_3\left(t-u\right) q\left(1,3,\mathrm{d}u\right)$$
$$= e^{-\left(\overline{\lambda}_A+\overline{\lambda}_B\right)t} + \int_0^t \left(\overline{\lambda}_B A_2\left(t-u\right) + \overline{\lambda}_A A_3\left(t-u\right)\right) e^{-\left(\overline{\lambda}_A+\overline{\lambda}_B\right)u} \mathrm{d}u,$$
$$A_2\left(t\right) = \int_0^t A_1\left(t-u\right) q\left(2,1,\mathrm{d}u\right) + \int_0^t A_3\left(t-u\right) q\left(2,3,\mathrm{d}u\right)$$
$$= \int_0^t \left(p_1 A_1\left(t-u\right) + \left(1-p_1\right) A_2\left(t-u\right)\right) f_{\mu_{\mathrm{B}}}\left(u\right) \mathrm{d}u,$$

with

$$p_1 = \mathbb{P}\left(Z_{\mu_{\mathrm{B}}} \le Z_{\overline{\lambda}'_A}\right) = \int_0^{+\infty} f_{\mu_{\mathrm{B}}}\left(t\right) e^{-\overline{\lambda}'_A t} \mathrm{d}t$$
$$= \frac{1}{\left(1 + b_{R_B} \overline{\lambda}'_A\right)^{a_{R_B}}},$$

and a similar equation for $A_2\left(t\right)$.

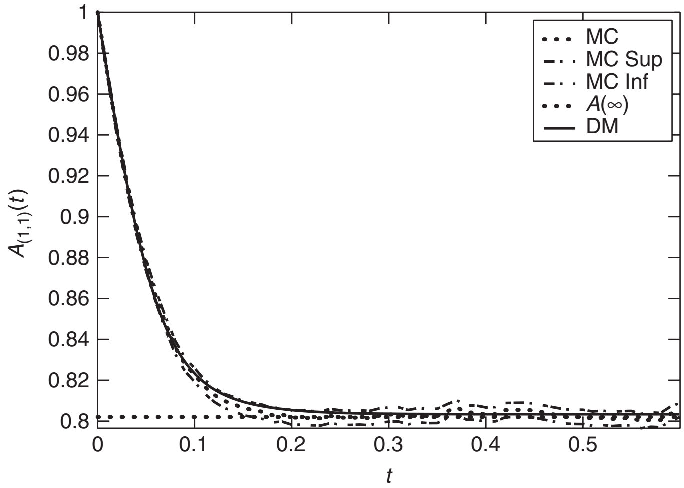

Figure 5. Point and asymptotic availabilities, Markov regenerative case.

This provides a set of three equations, which we discretize for their numerical resolution. Monte Carlo simulations are also performed. The results for $A_{(1,1)}(t)$ $(= A_1(t))$ are provided by both methods in Fig. 5, as well as the asymptotic availability, easily provided by Equation (15) with $A(\infty) \simeq 0.802$.

DISCUSSION AND FURTHER READING

In conclusion, we have presented here, classical stochastic models with (Markov) regenerative properties. For such models, the point availability has been proved to satisfy (Markov) renewal equations. The solving of such equations has, however, been seen to be generally impossible in full form and requires numerical procedures. We have here made the choice to use discretization techniques, which are easy to implement. Also, they can provide upper and lower bounds for the solutions. The results have been checked through Monte Carlo simulations, which are commonly used by practitioners in the reliability field, with generally longer computing times, however. Another numerical method might have been to use Laplace transforms, which are available in full form for the quantities of interest. The great progress made in their inversion in the last decade makes this method very appealing. Numerical difficulties may however arise in case of small or big arguments. Also, the precision of the results is not always available. See Ref. 17 for more details and references on the subject.

As for the asymptotic availability, it is usually simpler to compute than the point availability: it typically requires to evaluate the mean cycle duration and the mean up time on a cycle in the regenerative case, or similar quantities linked to the underlying Markov renewal process in the Markov regenerative case.

The classical models presented here actually sometimes (often?) appear as restrictive in the applications and even very simple systems may not meet with their assumptions: as an example, let us come back to our two-unit series system, where one component is suspended when the other is down, with general repair and failure rates and only down components repaired. In that case, the system never forgets its past so that the system meets with none of the previous models. Monte Carlo simulations may however be performed, to compute both point and asymptotic availabilities. Another possibility is to use new models coming from dynamic reliability [18], which are presently arriving in "classical" reliability. Such models are called *piecewise deterministic Markov processes* [19] and have been proved to have a great modeling power [20]. Like the models presented here, their numerical assessment may be done by Monte Carlo simulations or by discretization methods [21,22].

Another drawback of the models presented here is that no aging is taken into account in the sense that at (Markov) regeneration points, the future evolution of the system given its present state is a stochastic replica of the past given the same starting state, without any evolution. Other possible models, which take aging into account are geometric processes, where up and down periods alternate with geometrically decreasing and increasing durations [23], nonhomogeneous Markov and semi-Markov processes with finite state spaces [24], or nonhomogeneous piecewise deterministic Markov processes [25].

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